Projekt C

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Hold 2

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1 Opgave 1

a

Bestemmer en QR-faktorisering ved at benytte Gram-Schmidt processen:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -5 \\ 5 & -2 & -13 \\ -3 & 3 & 15 \\ 1 & -1 & 7 \end{bmatrix}$$

Først bestemmes søjlevektorerne $\mathbf{u}$ fra $\mathbf{A}$:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 5 \\ -3 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 3 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -5 \\ -13 \\ 15 \\ 7 \end{bmatrix}$$

Herefter bestemmes $\mathbf{q}$ vektorerne og r-værdierne:
$\mathbf{q}_1$:

$$r_{11} = ||\mathbf{u_1}|| = \sqrt{1^2 + 5^2 + (-3)^2 + 1^2} = 6$$

$$\mathbf{q}_1 = \frac{\mathbf{u_1}}{r_{11}} = \frac{1}{6}\begin{bmatrix} 1 \\ 5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \\ -\frac{3}{6} \\ \frac{1}{6} \end{bmatrix}$$

$\mathbf{q}_2$:

$$r_{12} = \mathbf{u}_2 \cdot \mathbf{q}_1 = \begin{bmatrix} 2 \\ -2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \\ -\frac{3}{6} \\ \frac{1}{6} \end{bmatrix} = 2 \cdot \frac{1}{6} - 2 \cdot \frac{5}{6} - 3 \cdot \frac{3}{6} - 1 \cdot \frac{1}{6} = -3$$

$$r_{22} = \|\mathbf{u}_2 - r_{12}\mathbf{q}_1\| = \| \begin{bmatrix} 2 \\ -2 \\ 3 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \\ -\frac{3}{6} \\ \frac{1}{6} \end{bmatrix} \| = \| \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} \| =$$

$$\sqrt{(\frac{5}{2})^2 + (\frac{1}{2})^2 + (\frac{3}{2})^2 + (\frac{1}{2})^2} = 3$$

$$\mathbf{q}_2 = \frac{\mathbf{u}_2 - r_{12}\mathbf{q}_1}{r_{22}} = \frac{1}{3} \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{1}{6} \\ \frac{1}{2} \\ -\frac{1}{6} \end{bmatrix}$$

$\mathbf{q}_3$:

$$r_{13} = \mathbf{u}_3 \cdot \mathbf{q}_1 = \begin{bmatrix} -5 \\ -13 \\ 15 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \\ -\frac{3}{6} \\ \frac{1}{6} \end{bmatrix} = -5 \cdot \frac{1}{6} - 13 \cdot \frac{5}{6} - 15 \cdot \frac{3}{6} + 7 \cdot \frac{1}{6} = -18$$

$$r_{23} = \mathbf{u}_3 \cdot \mathbf{q}_2 = \begin{bmatrix} -5 \\ -13 \\ 15 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} \frac{5}{6} \\ \frac{1}{6} \\ \frac{1}{2} \\ -\frac{1}{6} \end{bmatrix} = -5 \cdot \frac{5}{6} - 13 \cdot \frac{1}{6} + 15 \cdot \frac{1}{2} - 7 \cdot \frac{1}{6} = 0$$

$$r_{33} = \|\mathbf{u}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2\| = \|\mathbf{u}_3 - r_{13}\mathbf{q}_1\| = \| \begin{bmatrix} -5 \\ -13 \\ 15 \\ 7 \end{bmatrix} + 18 \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \\ -\frac{3}{6} \\ \frac{1}{6} \end{bmatrix} \| = \| \begin{bmatrix} -2 \\ 2 \\ 6 \\ 10 \end{bmatrix} \| =$$

$$\sqrt{(-2)^2 + 2^2 + 6^2 + 10^2} = 12$$

$$\mathbf{q}_3 = \frac{\mathbf{u}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2}{r_{33}} = \frac{1}{12} \begin{bmatrix} -2 \\ 2 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{2} \\ \frac{10}{12} \end{bmatrix}$$

Og derfra kan matricerne $\mathbf{Q}$ og $\mathbf{R}$ opstilles:

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 6 & -3 & -18 \\ 0 & 3 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

b

Projektionsmatricen $\mathbf{P}$ er bestemt ved $\mathbf{P} = \mathbf{Q}\mathbf{Q}^\top$ hvor

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

og

$$\mathbf{Q}^\top = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{2} & \frac{5}{6} \end{bmatrix}$$

Hvor matrix produktet kan udregnes ved brug af python funktionen fra projekt A:

$$\mathbf{P} = \mathbf{Q}\mathbf{Q}^\top = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{2} & \frac{5}{6} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix}$$

c

Den ortogonale projektion af $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ på underrummet $\mathcal{U}$:

$$\text{proj}_{\mathcal{U}}(\mathbf{v}) = \mathbf{P}\mathbf{v} = \frac{1}{4} \begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Det vides at spejlingen $\text{refl}_{\mathcal{U}}(\mathbf{v}) = \mathbf{Rv}$ hvor $\mathbf{R} = 2\mathbf{P} - \mathbf{I}$:

$$\mathbf{R} = 2\frac{1}{4}\begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\text{refl}_U(\mathbf{v}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

d

Det vides at $\mathcal{U}^\perp = (\text{col}\mathbf{A})^\perp = \text{null}(\mathbf{A}^T)$, bestemmer derfor $\text{null}(\mathbf{A}^T)$:

$$\mathbf{A}^T = \begin{bmatrix} 1 & 5 & -3 & 1 \\ 2 & -2 & 3 & -1 \\ -5 & -13 & 15 & 7 \end{bmatrix}$$

Løser herefter $\mathbf{Ax} = 0$:

$$\left[\begin{array}{cccc|c} 1 & 5 & -3 & 1 & 0 \\ 2 & -2 & 3 & -1 & 0 \\ -5 & -13 & 15 & 7 & 0 \end{array}\right] -2r_1 + r_2 \to r_2 \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 5 & -3 & 1 & 0 \\ 0 & -12 & 9 & -3 & 0 \\ -5 & -13 & 15 & 7 & 0 \end{array}\right]$$

$$5r_1 + r_3 \to r_3 \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 5 & -3 & 1 & 0 \\ 0 & -12 & 9 & -3 & 0 \\ 0 & 12 & 0 & 12 & 0 \end{array}\right] r_2 + r_3 \to r_3 \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 5 & -3 & 1 & 0 \\ 0 & -12 & 9 & -3 & 0 \\ 0 & 0 & 9 & 9 & 0 \end{array}\right]$$

$$-\frac{1}{12}r_2 \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 5 & -3 & 1 & 0 \\ 0 & 1 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 9 & 9 & 0 \end{array}\right] \frac{1}{9}r_3 \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 5 & -3 & 1 & 0 \\ 0 & 1 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\frac{3}{4}r_3 + r_2 \to r_2 \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 5 & -3 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] 3r_3 + r_1 \to r_1 \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 5 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$-5r_2 + r_1 \to r_1 \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Her ses det, at x_4 er en fri variable, som sættes $x_4 = t$, og løsningerne kan angives:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Og derfra ses det, at en basis til underummet $\mathcal{U}^{\perp}$: $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

e

Først opstilles matricen **B** ved at tilføje den normerede vektor i fjerde kolonne fra opgave **d**:

$$\| \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \| = \sqrt{1^2 + (-1)^2 + (-1)^2 + 1^2} = 2$$

Den normerede vektor er derfor

$$\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Opstiller **B**:

$$\mathbf{B} = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & -\frac{1}{6} & \frac{1}{2} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{6} & \frac{5}{6} & \frac{1}{2} \end{bmatrix}$$

Hvis matricen er ortonormal er den transponerede den inverse (Definition 4.9):

$$\mathbf{B^T B} = \mathbf{I}$$

$$\mathbf{B^T} = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{2} & \frac{5}{6} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{B^T B} = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{2} & \frac{5}{6} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & -\frac{1}{6} & \frac{1}{2} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{6} & \frac{5}{6} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Da er $\mathbf{B}^{-1}$ bestemt ved

$$\mathbf{B}^{-1} = \mathbf{B^T} = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{2} & \frac{5}{6} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Fra Theorem 4.7 vides det, at når en ortonormal matrice ganges på en vektor, beholder vektoren sin længde. Derfor behøves der blot at udregne længden af $\mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{\sqrt{2}^2 + \sqrt{3}^2 + \sqrt{5}^2 + \sqrt{6}^2} = 4$$

2

a

Det karakteristiske polynomie er givet ved

$$p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$$

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 2 & 1 \\ -2 & 3 & 1 \\ -2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} \lambda+1 & -2 & -1 \\ 2 & \lambda-3 & -1 \\ 2 & -2 & \lambda-2 \end{bmatrix}$$

Udfører herefter 2 rækkeoperationer på matricen:

$$\begin{bmatrix} \lambda+1 & -2 & -1 \\ 2 & \lambda-3 & -1 \\ 2 & -2 & \lambda-2 \end{bmatrix} - r_2 + r_1 \to r_1 \rightsquigarrow \begin{bmatrix} \lambda-1 & -\lambda+1 & 0 \\ 2 & \lambda-3 & -1 \\ 2 & -2 & \lambda-2 \end{bmatrix}$$

$$-r_2 + r_3 \to r_3 \begin{bmatrix} \lambda-1 & -\lambda+1 & 0 \\ 2 & \lambda-3 & -1 \\ 0 & -\lambda+1 & \lambda-1 \end{bmatrix}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det\left(\begin{bmatrix} \lambda-1 & -\lambda+1 & 0 \\ 2 & \lambda-3 & -1 \\ 0 & -\lambda+1 & \lambda-1 \end{bmatrix}\right) =$$

$$(\lambda-1)^2\det\left(\begin{bmatrix}1 & -1 & 0\\ 2 & \lambda-3 & -1\\ 0 & -1 & 1\end{bmatrix}\right)$$

Udvikler herefter efter første række for at bestemme det karakteristiske polynomie:

$$(\lambda-1)^2\left(1\cdot\det\left(\begin{bmatrix}\lambda-3 & -1\\ -1 & 1\end{bmatrix}\right)+(-1)^3\det\left(\begin{bmatrix}2 & -1\\ 0 & 1\end{bmatrix}\right)+0\cdot\det\left(\begin{bmatrix}2 & \lambda-3\\ 0 & -1\end{bmatrix}\right)\right)=$$

$$(\lambda-1)^2(\lambda-3-1+2)=(\lambda-1)^2(\lambda-2)$$

Derfra ses det, at det karakteristiske polynomie er $(\lambda-1)^2(\lambda-2)$

b

Fra opgave **a** ses det, at egenværdierne er $\lambda=1$ & $\lambda=2$ hvor $\lambda=1$ har en algebraisk multiplicitet på 2 og $\lambda=2$ har en algebraisk multiplicitet på 1.

c

En basis for egenrummet kan bestemmes ved $E_\lambda=\text{null}(\mathbf{A}-\lambda\mathbf{I})$.
En basis for E_1:

$$E_1=\text{null}(\mathbf{A}-1\cdot\mathbf{I})=\text{null}\left(\begin{bmatrix}-1 & 2 & 1\\ -2 & 3 & 1\\ -2 & 2 & 2\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)=$$

$$\text{null}\left(\begin{bmatrix}-2 & 2 & 1\\ -2 & 2 & 1\\ -2 & 2 & 1\end{bmatrix}\right)$$

Benytter Gaus-Jordan eliminition for at bestemme null-rummet:

$$\begin{bmatrix}-2 & 2 & 1\\ -2 & 2 & 1\\ -2 & 2 & 1\end{bmatrix}\rightsquigarrow\begin{bmatrix}1 & -1 & -\frac{1}{2}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Derfra kan løsningerne bestemmes, hvor $x_2=s$ og $x_3=t$:

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=s\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}\frac{1}{2}\\ 0\\ 1\end{bmatrix}$$

Derfra ses det, at en basis for E_1:$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}\frac{1}{2}\\0\\1\end{bmatrix}\right\}$

En basis for E_2:

$$E_2 = E_1 = \text{null}(\mathbf{A} - 2\cdot\mathbf{I}) = \text{null}\left(\begin{bmatrix}-1 & 2 & 1\\-2 & 3 & 1\\-2 & 2 & 2\end{bmatrix} - \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\right) =$$

$$\text{null}\left(\begin{bmatrix}-3 & 2 & 1\\-2 & 1 & 1\\-2 & 2 & 0\end{bmatrix}\right)$$

Benytter Gaus-Jordan eliminition for at bestemme null-rummet:

$$\begin{bmatrix}-3 & 2 & 1\\-2 & 1 & 1\\-2 & 2 & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & -1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

Derfra kan løsningerne opskrives, hvor $x_3 = t$:

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = t\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Derfra ses det, at en basis for E_2:$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

For E_1 ses det, at der er to frie variable, og har derfor en geometrisk multiplicitet på 2. For E_2 ses det, at der er en frie variable, og har derfor en geometrisk multiplicitet på 1.

d

P bestemmes blot ved at danne en matrice ud fra vektorerne i egenrummende:

$$\mathbf{P} = \begin{bmatrix}1 & \frac{1}{2} & 1\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$$

$\mathbf{P}^{-1}$ kan herefter bestemmes:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] r_2 \leftrightarrow r_3 \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] - r_1 + r_3 \to r_3 \rightsquigarrow r_3$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 & -1 & 1 & 0 \end{array}\right] - 2r_3 \to r_3 \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & -2 & 0 \end{array}\right]$$

$$r_2 \leftrightarrow r_3 \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] - r_2 + r_3 \to r_3 \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right]$$

$$-\frac{1}{2}r_2 + r_1 \to r_1 \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right] - r_3 + r_1 \to r_1 \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right]$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -2 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

Og matricen $\mathbf{D}$ bestemmes ved at indsætte egenværdierne på diagonallen gentaget henholdsvis 2 og 1 gang efter deres geometriske multiplicitet:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Herefter kan $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ kontrolleres:

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -2 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ -2 & 3 & 1 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \mathbf{D}$$

e

Et generelt udtryk for $\mathbf{A}^n$ kan gives ved (givet i slides):

$$\mathbf{A}^n = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$$

Matricen kan derfor bestemmes til:

$$\mathbf{A}^n = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^n \begin{bmatrix} 2 & -1 & -1 \\ 2 & -2 & 0 \\ -2 & 2 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 3 - 2 \cdot 2^n & -2 + 2 \cdot 2^n & -1 + 2^n \\ 2 - 2 \cdot 2^n & -1 + 2 \cdot 2^n & -1 + 2^n \\ 2 - 2 \cdot 2^n & -2 + 2 \cdot 2^n & 2^n \end{bmatrix}$$

Opgave 3

Opstiller en matrice ved brug af mindste kvadraters metode ift. ligningen $c + xt = y$, hvor $t = 0$ i 2010:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 6 \\ 1 & 8 \\ 1 & 10 \\ 1 & 12 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} b \\ a \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 35,481 \\ 36,891 \\ 37,331 \\ 38,061 \\ 39,071 \\ 39,345 \\ 40,568 \\ 42,140 \end{bmatrix}$$

Løsningen er derfra givet ved $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$, som udregnes i python til:

$$(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} = \begin{bmatrix} 36,1436909090909 \\ 0,469963636363636 \end{bmatrix}$$

Dvs. en forskrift for den rette linje er

$$ln(y) = 0,469963636363636t + 36,1436909090909$$

b

Omskriver funktionen fundet i opgave **a**:

$$ln(y) = 0,469963636363636t + 36,1436909090909$$

$$\iff$$

$$e^{ln(y)} = e^{0,469963636363636t+36,1436909090909}$$

$$\iff$$

$$y = e^{0,469963636363636t} \cdot e^{36,1436909090909}$$

$$\iff$$

$$y = 4,977434089 \cdot 10^{15} \cdot e^{0,469963636363636t}$$

Herefter vides det, at i vores metode satte vi $t = 0$ i 2010, så derfor omskrives det til t til $t - 2010$ og afrunder:

$$y = 4,98 \cdot 10^{15} \cdot e^{0,47(t-2010)}$$

c

Sætter $t = 2000$ i $y(t)$:

$$y(2010) = 4,98 \cdot 10^{15} \cdot e^{0,470(2000-2010)} = 4,529447997 \cdot 10^{13}$$

Dvs. i år 2000 vil verdens bedste super computer kunne præstere $4,529447997 \cdot 10^{13}$ FLOPS.
Sætter $t = 2030$ for at bestemme antal FLOPS i verdens bedste super computer i år 2030:

$$y(2010) = 4,98 \cdot 10^{15} \cdot e^{0,470(2030-2010)} = 6,020013604 \cdot 10^{19}$$